

# COLORING NON-CROSSING STRINGS

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**ABSTRACT.** For a family of geometric objects in the plane  $\mathcal{F} = \{S_1, \dots, S_n\}$ , define  $\chi(\mathcal{F})$  as the least integer  $\ell$  such that the elements of  $\mathcal{F}$  can be colored with  $\ell$  colors, in such a way that any two intersecting objects have distinct colors. When  $\mathcal{F}$  is a set of pseudo-disks that may only intersect on their boundaries, and such that any point of the plane is contained in at most  $k$  pseudo-disks, it can be proven that  $\chi(\mathcal{F}) \leq 3k/2 + o(k)$  since the problem is equivalent to cyclic coloring of plane graphs. In this paper, we study the same problem when pseudo-disks are replaced by a family  $\mathcal{F}$  of pseudo-segments (a.k.a. strings) that do not cross. In other words, any two strings of  $\mathcal{F}$  are only allowed to “touch” each other. Such a family is said to be  $k$ -touching if no point of the plane is contained in more than  $k$  elements of  $\mathcal{F}$ . We give bounds on  $\chi(\mathcal{F})$  as a function of  $k$ , and in particular we show that  $k$ -touching segments can be colored with  $k + 5$  colors. This partially answers a question of Hliněný (1998) on the chromatic number of contact systems of strings.

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## 1. INTRODUCTION

For a family  $\mathcal{F} = \{S_1, \dots, S_n\}$  of subsets of a set  $\Omega$ , the *intersection graph*  $G(\mathcal{F})$  of  $\mathcal{F}$  is defined as the graph with vertex-set  $\mathcal{F}$ , in which two vertices are adjacent if and only if the corresponding sets have non-empty intersection.

For a graph  $G$ , the *chromatic number* of  $G$ , denoted  $\chi(G)$ , is the least number of colors needed in a proper coloring of  $G$  (a coloring such that any two adjacent vertices have distinct colors). When talking about a proper coloring of a family  $\mathcal{F}$  of subsets of a given set, we implicitly refer to a proper coloring of the intersection graph of  $\mathcal{F}$ , thus the chromatic number  $\chi(\mathcal{F})$  is defined in a natural way.

The chromatic number of families of geometric objects in the plane have been extensively studied since the sixties [2, 12, 15, 16, 17]. Since it is possible to construct sets of pairwise intersecting (straight-line) segments of any size, the chromatic number of sets of segments in the plane is unbounded in general. However, Erdős conjectured that triangle-free intersection graphs of segments in the plane have bounded chromatic number (see [11]). This was recently disproved [20]. The conjecture of Erdős initiated the study of the chromatic number of families of geometric objects in the plane as a function of their *clique number*, the maximum size of subsets of the family that pairwise intersect [7]. In this paper, we consider families of geometric objects in the plane for which the chromatic number only depends on local properties of the families, such as the maximum number of objects containing a given point of the plane.

Consider a set  $\mathcal{F} = \{\mathcal{R}_1, \dots, \mathcal{R}_n\}$  of pseudo-disks (subsets of the plane which are homeomorphic to a closed disk) such that the intersection of the interiors of any two pseudo-disks is empty. Let  $\mathcal{H}_{\mathcal{F}}$  be the planar hypergraph with vertex set  $\mathcal{F}$ , in which the hyperedges are the maximal sets of pseudo-disks whose intersection is non-empty. A proper coloring of  $\mathcal{F}$  is equivalent to a coloring of  $\mathcal{H}_{\mathcal{F}}$  in which all the vertices of each hyperedge have distinct colors. If every point is contained in at most  $k$  pseudo-disks, Borodin conjectured that there exists such a coloring of  $\mathcal{H}_{\mathcal{F}}$  with at most  $\frac{3}{2}k$  colors [3]. It was recently proven that this conjecture holds asymptotically [1] (not only in the plane, but also on any fixed surface). As a consequence,  $\mathcal{F}$  can be properly colored with  $\frac{3}{2}k + o(k)$  colors.

It seems natural to investigate the same problem when pseudo-disks are replaced by *pseudo-segments*. These are continuous injective functions from  $[0, 1]$  to  $\mathbb{R}^2$  and are usually referred to as *strings*. Consider a set  $\mathcal{S} = \{C_1, \dots, C_n\}$  of such strings. We say that  $\mathcal{S}$  is *touching* if no pair of strings of  $\mathcal{S}$  cross, and that it is *k-touching* if furthermore at most  $k$  strings can “touch” in any point of the plane, i.e., any point of the plane is contained in at most  $k$  strings (see Figure 1(a) for an example).

Note that the family of all touching sets of strings contains all *contact systems of strings*, defined as sets of strings such that the interior of any

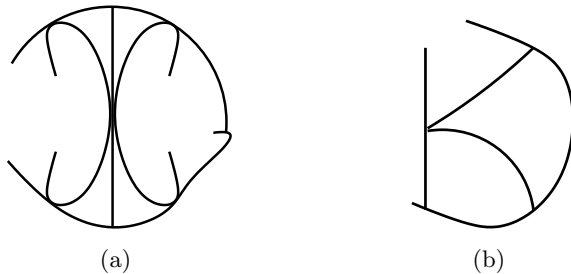


FIGURE 1. (a) A 3-touching set of strings  $\mathcal{S}_1$  with  $G(\mathcal{S}_1) \cong K_5$ . (b) A one-sided 3-contact representation of curves  $\mathcal{S}_2$  with  $G(\mathcal{S}_2) \cong K_4$ .

two strings have empty intersection. In other words, if  $c$  is a contact point in the interior of a string  $s$ , all the strings containing  $c$  distinct from  $s$  end at  $c$ . In [13], Hliněný studied contact system of strings such that all the strings ending at  $c$  leave from the same side of  $s$ . Such a representation is said to be *one-sided* (see Figure 1(b) for an example). It was proved in [13] that if a contact system of strings is  $k$ -touching and every contact point is one-sided, then the strings can be colored with  $2k$  colors.

In this paper, our aim is to study  $k$ -touching sets of strings in their full generality. Observe that if  $\mathcal{S}$  is  $k$ -touching,  $k$  might be much smaller than the maximum degree of  $\mathcal{S}$ . However, based on the cases of pseudo-disks and contact system of strings, we conjectured the following in the conference version of this paper [5]:

**Conjecture 1.1.** *For some constant  $c > 0$ , any  $k$ -touching set of strings can be colored with  $ck$  colors.*

This conjecture was subsequently proved by Fox and Pach [8], who showed that any  $k$ -touching set of strings can be colored with  $6ek + 1$  colors (where  $e$  is the base of the natural logarithm). In Section 2, we show how to slightly improve their bound for small values of  $k$ . We also show that for any odd  $k$ , the clique on  $\frac{9}{2}(k - 1)$  vertices can be represented as a set of  $k$ -touching strings, so the best possible constant  $c$  in Conjecture 1.1 is between 4.5 and  $6e \approx 16.3$ .

In Section 3, we give improved bounds when any two strings can intersect a bounded number of times. In Section 4, we restrict ourselves to contact systems of strings where any two strings intersect at most once (called 1-intersecting, which were previously studied by Hliněný [13]). He asked whether there is a constant  $c$  such that every one-sided 1-intersecting  $k$ -touching contact system of strings is  $(k + c)$ -colorable. We prove that they are  $(\frac{4k}{3} + 6)$ -colorable, and that every  $k$ -touching contact system of segments is  $(k + 5)$ -colorable. Note that we do not need our contact systems to be one-sided

(but adding this assumption offers a minor improvement on the additive constants).

Before giving general bounds, let us first mention two classical families of touching strings for which coloring problems are well understood.

If a  $k$ -touching set of strings has the property that the interior of each string is disjoint from all the other strings, then each string can be thought of as an edge of some (planar) graph with maximum degree  $k$ . By a classical theorem of Shannon, the strings can then be colored with  $3k/2$  colors. If moreover, any two strings intersect at most once, then they can be colored with  $k+1$  colors by a theorem of Vizing (even with  $k$  colors whenever  $k \geq 7$ , using a more recent result of Sanders and Zhao [21]). In this sense, all the problems considered in this article can be seen as an extension of edge-coloring of planar graphs.

An *x-monotone string* is a string such that every vertical line intersects it in at most one point. Alternatively, it can be defined as the curve of a continuous function from an interval of  $\mathbb{R}$  to  $\mathbb{R}$ . Sets of  $k$ -touching *x-monotone strings* are closely related to bar  $k$ -visibility graphs. A *bar  $k$ -visibility graph* is a graph whose vertex-set consists of horizontal segments in the plane (bars), and two vertices are adjacent if *and only if* there is a vertical segment connecting the two corresponding bars, and intersecting no more than  $k$  other bars. It is not difficult to see that the graph of any set of  $k$ -touching *x-monotone strings* is a spanning subgraph of some bar  $(k-2)$ -visibility graph, while any bar  $(k-2)$ -visibility graph can be represented as a set of  $k$ -touching *x-monotone strings*. Using this correspondence, it directly follows from [4] that  $k$ -touching *x-monotone strings* are  $6k-6$  colorable, and that the clique on  $4k-4$  vertices can be represented as a set of  $k$ -touching *x-monotone strings*. If the left-most point of each *x-monotone string* intersects the vertical line  $x=0$ , then it directly follows from [6] that the strings can be colored with  $2k-1$  colors (and a clique on  $2k-1$  vertices can be represented by  $k$ -touching *x-monotone strings* in this specific way).

## 2. GENERAL BOUNDS

Before proving any result on the structure of sets of  $k$ -touching strings in general, we make the following observation:

**Observation 2.1.** *The family of intersection graphs of 2-touching strings is exactly the class of planar graphs.*

The class of planar graphs being exactly the class of intersection graphs of 2-touching pseudo-disks (see [14]) it is clear that planar graphs are intersection graphs of 2-touching strings (by taking a connected subset of the boundaries of each pseudo-disk). Furthermore, every intersection graph of 2-touching strings is contained in the intersection graph of 2-touching pseudo-disks, and is thus planar. Indeed, it is easy given a set of 2-touching strings

$\mathcal{S} = \{\mathcal{C}_1, \dots, \mathcal{C}_n\}$  to draw a set of 2-touching pseudo-disks  $\mathcal{F} = \{\mathcal{R}_1, \dots, \mathcal{R}_n\}$  such that  $\mathcal{C}_i \subset \mathcal{R}_i$  for every  $i \in [1, n]$ .

The following result was proved by Fox and Pach [8] (this shows Conjecture 1.1). In the following,  $e$  is the base of the natural logarithm.

**Theorem 2.2** ([8]). *Any  $k$ -touching set of strings is  $(6ek + 1)$ -colorable.*

Their theorem is a direct consequence on the following bound on the number of edges in a graph represented by a set of  $k$ -touching strings. Their proof is inspired by the probabilistic proof of the Crossing Lemma.

**Lemma 2.3** ([8]). *Any graph represented by a set of  $n$   $k$ -touching strings has less than  $3ekn$  edges.*

When  $k = 3$ , their proof can easily be optimized to show that the number of edges is less than  $12n$ . Hence, every such graph has a vertex with degree less than 24. These graphs are thus 23-degenerate and have chromatic number at most 24. We now show how to modify their proof to slightly improve this bound.

**Theorem 2.4.** *Any 3-touching set of strings is 19-colorable.*

This is a direct consequence of the following lemma.

**Lemma 2.5.** *Any graph represented by a set of  $n$  3-touching strings has at most  $\frac{6}{7}(6 + \sqrt{22})n \approx 9.16n$  edges.*

*Proof.* Our proof follows the same lines as that of Lemma 2.3 in [8]. Let  $\mathcal{S}$  be a set of  $n$  3-touching strings, and let  $m$  be the number of edges in the corresponding intersection graph  $G$ . We can assume that for any point  $p$  contained in three different strings, one of the strings is “sandwiched” between the other two (see Figure 3, left). For otherwise, we can apply the local transformation described in Figure 2 and turn  $p$  into three different points, each containing exactly two strings, without changing  $G$ .

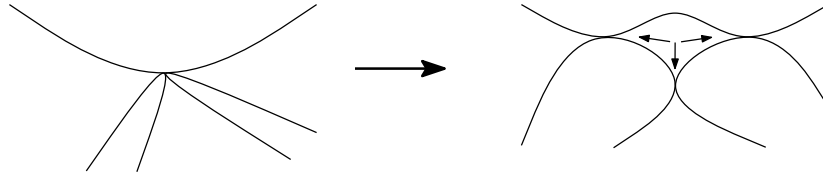


FIGURE 2. A local modification turning one point into three points, without changing the intersection graph.

We now define two spanning subgraphs  $G_1$  and  $G_0$  of  $G$  as follows. Two strings  $a$  and  $c$  are adjacent in  $G_1$  if they intersect and for every intersection point  $p$  of  $a$  and  $c$ , there exists a third string  $b$  that is sandwiched between  $a$  and  $c$ . For every pair of strings  $a$  and  $c$  adjacent in  $G_1$ , let  $P_1(a, c)$  be

an arbitrarily chosen intersection point between  $a$  and  $c$  (with some third string sandwiched between  $a$  and  $c$ ).

Two strings  $a$  and  $c$  are adjacent in  $G_0$  if they are adjacent in  $G$  but not in  $G_1$ , i.e., if there exists an intersection point  $P_0(a, c)$  of  $a$  and  $c$  such that either  $P_0(a, c)$  is not contained in another string or  $P_0(a, c)$  is contained in a third string  $b$  that is not sandwiched between  $a$  and  $c$ . For  $i = 0, 1$ , the edge-set of  $G_i$  is denoted by  $E_i$ , and the cardinality of  $E_i$  is denoted by  $m_i$ .

**Claim 2.6.**  $7m_0 \geq m + 6n$

For each edge  $ab \in E_1$ , assume that  $ab$  gives a charge of 1 to the string  $c$  sandwiched between  $a$  and  $b$  at  $P_1(a, b)$ . Let  $N_0(c)$  be the set of neighbors of  $c$  in  $G_0$ . The total charge  $\rho(c)$  received by  $c$  is at most the number of pairs of vertices  $x, y \in N_0(c)$  such that in  $\mathcal{S} \setminus \{c\}$ ,  $P_0(x, y)$  is a 2-contact point. If we modify the set of strings intersecting  $c$  so as to only preserve those 2-touching points (all the other pairs of strings are made disjoint), we obtain a planar graph with vertex-set  $N_0(c)$  and with  $\rho(c)$  edges. It follows that  $\rho(c) \leq 3|N_0(c)| - 6$ . Summing for all strings  $c$ , we obtain that the total charge  $m_1 = \sum_{c \in \mathcal{S}} \rho(c) \leq 6m_0 - 6n$ . Since  $m_0 + m_1 = m$ , we have  $7m_0 \geq m + 6n$ , as claimed.

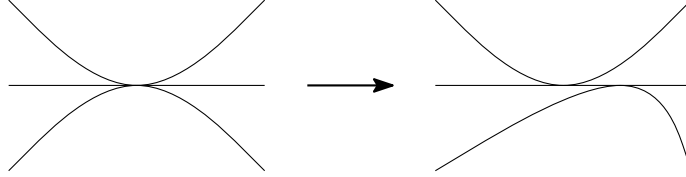


FIGURE 3. A local modification preserving adjacency in the graph  $G_0$ .

We are now ready to prove the lemma. We select each string of  $\mathcal{S}$  uniformly at random with probability  $p$  (to be chosen later). In the subset of chosen strings, we slightly modify each 3-touching point as depicted in Figure 3 in order to preserve all edges in  $G_0$ . Let  $\mathcal{S}'$  be the new set of strings. For each edge  $ab \in E_0$ , the probability that  $ab$  appears as an edge in  $\mathcal{S}'$  is  $p^2$ , and for each edge  $ab \in E_1$ , the probability that  $ab$  appears as an edge in  $\mathcal{S}'$  is at least the probability that  $a$  and  $b$  were both selected and the string  $c$  sandwiched between  $a$  and  $b$  at  $P_1(a, b)$  was not selected, which is  $p^2(1 - p)$ . The expected number of vertices of the graph represented by  $\mathcal{S}'$  is then  $pn$  and its expected number of edges is at least

$$\begin{aligned}
 m_0 p^2 + m_1 p^2 (1 - p) &= p^2 (m_0 + m_1 (1 - p)) \\
 &= p^2 (m_0 p + m (1 - p)) \\
 &\geq \frac{p^2}{7} ((m + 6n)p + 7m(1 - p)) \\
 &\geq \frac{p^2}{7} (m(7 - 6p) + 6pn)
 \end{aligned}$$

By Observation 2.1, the graph is planar and therefore,  $\frac{p^2}{7} (m(7 - 6p) + 6pn) \leq 3pn$ . This can be rewritten as  $m \leq n \frac{21 - 6p^2}{p(7 - 6p)}$ . Taking  $p = 3 - \sqrt{11/2}$ , we obtain  $m \leq \frac{6}{7} (6 + \sqrt{22})n$ .  $\square$

The ideas of Lemma 2.5 can be used to slightly improve the multiplicative constant in Theorem 2.2 for all  $k$ . Since our improvement is minor (we obtain a bound of  $6k \times 2.686$  instead of  $6k \times 2.718$ ), we omit the details.

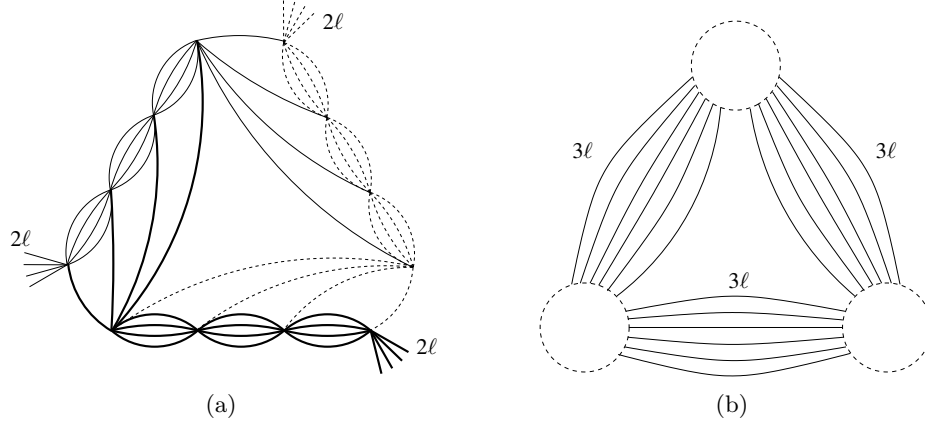


FIGURE 4. (a) The construction of a  $2\ell$ -sun. (b) A set  $\mathcal{S}$  of  $k$ -touching strings requiring  $\lceil \frac{9k}{2} \rceil - 5$  distinct colors.

We now show that the constant  $c$  in Conjecture 1.1 is at least  $\frac{9}{2}$ .

**Theorem 2.7.** *For every odd  $k \geq 1$ ,  $k = 2\ell + 1$ , there exists a set of  $k$ -touching strings  $\mathcal{S}_k$  such that the strings of  $\mathcal{S}_k$  pairwise touch and such that  $|\mathcal{S}_k| = 9\ell = 9(k - 1)/2$ . Thus  $\chi(\mathcal{S}_k) = 9\ell = \lceil \frac{9k}{2} \rceil - 5$ .*

Consider  $n$  touching strings  $s_1, \dots, s_n$  that all intersect  $n$  points  $c_1, \dots, c_n$  in the same order (see the set of bold strings in Figure 4(a) for an example when  $n = 4$ ), and call this set of strings an  $n$ -braid. For some  $\ell > 0$ , take three  $2\ell$ -braids  $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ , and for  $i = 1, 2, 3$ , connect each of the strings of  $\mathcal{S}_i$  to a different intersection point of  $\mathcal{S}_{i+1}$  (with indices taken modulo 3), while keeping the set of strings touching (see Figure 4(a)). We call this set of touching strings a  $2\ell$ -sun. Observe that a  $2\ell$ -sun contains  $6\ell$  strings (the

rays) that pairwise intersect, and that each intersection point contains at most  $2\ell + 1$  strings.

We now take three  $2\ell$ -suns and paste their respective rays as depicted in Figure 4(b). We obtain a  $(2\ell + 1)$ -touching set containing  $9\ell$  pairwise intersecting strings. Hence we need at least  $\lceil \frac{9k}{2} \rceil - 5$  colors in any proper coloring of this set of strings.

### 3. $\mu$ -INTERSECTING STRINGS

Let  $\mathcal{S}$  be a  $k$ -touching set of strings. The set  $\mathcal{S}$  is said to be  $\mu$ -*intersecting* if any two strings intersect in at most  $\mu$  points. We denote by  $H(\mathcal{S})$  the multigraph associated to  $\mathcal{S}$ : the vertices of  $H(\mathcal{S})$  are the strings of  $\mathcal{S}$ , and two strings with  $t$  common points correspond to two vertices connected by  $t$  edges in  $H(\mathcal{S})$ .

We prove the following result (which only supersedes Theorem 2.2 for  $\mu \leq 5$ ).

**Theorem 3.1.** *Any  $k$ -touching set  $\mathcal{S}$  of  $\mu$ -intersecting strings can be properly colored with  $3\mu k$  colors.*

Again, the proof is based on an upper bound on the number of edges of such graphs.

**Lemma 3.2.** *If  $\mathcal{S}$  is a  $k$ -touching set of  $n$   $\mu$ -intersecting strings, then  $H(\mathcal{S})$  has less than  $\frac{3}{2}\mu kn$  edges.*

*Proof.* For a string  $s$ , let  $d(s)$  denote the number of strings intersecting  $s$ , and let  $n$  denote the number of strings of  $\mathcal{S}$ . Let us denote by  $D(c)$  the number of strings containing an intersection point  $c$  (for any  $c$ ,  $D(c) \geq 2$  by definition), and let  $N$  denote the number of intersection points of  $\mathcal{S}$ . Note that  $k \geq D(c)$  for all intersection points  $c$ .

Let us slightly modify  $\mathcal{S}$  in order to obtain a set  $\mathcal{S}'$  of 2-touching and  $\mu$ -intersecting strings. For that, repeat the following operation while there exists an intersection point  $c$  with  $D(c) > 2$ . Consider a string  $s \ni c$  such that all the other strings at  $c$  are on the same side of  $s$  (such a string exists since strings do not cross each other). Move  $s$  along another string  $s' \ni c$  to a new intersection point  $c'$  in such way that the strings remain touching (see Figure 3 for the case  $k = 3$ ).

Each intersection point  $c$  in  $\mathcal{S}$  corresponds to a set  $X_c$  of intersection points in  $\mathcal{S}'$ . Let  $N'$  be the number of intersection points in  $\mathcal{S}'$ . By construction, each  $X_c$  has size exactly  $D(c) - 1$ , hence  $N' = \sum_c |X_c| = \sum_c (D(c) - 1)$ . Since the intersection graph of  $\mathcal{S}'$  is a planar graph with  $n$  vertices, it has at most  $3n - 6$  edges. As  $\mathcal{S}'$  is  $\mu$ -intersecting,  $N' \leq (3n - 6)\mu$ . As  $D(c) \leq k$  for any intersection point  $c$  in  $\mathcal{S}$ , we have

$$\sum_c D(c)(D(c) - 1) \leq kN' \leq (3n - 6)\mu k < 3\mu kn.$$



Finally, since the number of edges of  $H(\mathcal{S})$  is  $\frac{1}{2} \sum_{s \in \mathcal{S}} \sum_{c \in s} (D(c) - 1)$  which equals  $\frac{1}{2} \sum_c D(c)(D(c) - 1)$ , we have that  $H(\mathcal{S})$  has less than  $\frac{3}{2} \mu k n$  edges.  $\square$

In particular, if a  $k$ -touching set  $\mathcal{S}$  of strings is such that any two strings intersect in at most one point, Theorem 3.1 yields a bound of  $3k$  for the chromatic number of  $\mathcal{S}$ . We suspect that it is far from tight:

**Conjecture 3.3.** *There is a constant  $c > 0$ , such that every  $k$ -touching set of 1-intersecting strings can be colored with  $k + c$  colors.*

In the next section, we show that this conjecture holds for  $k$ -touching (straight-line) segments, a special case of 1-intersecting strings. It is interesting to note that even though the bound for  $k$ -touching  $\mu$ -intersecting graphs in Conjecture 1.1 and Theorem 2.2 does not depend on  $\mu$ , the chromatic number of these graphs has some connection with  $\mu$ : sets of strings with  $\mu = 1$  have chromatic number at most  $3k$ , whereas there exists sets of strings with large  $\mu$  and chromatic number at least  $\frac{9}{2}(k - 1)$ .



FIGURE 5. (a) A 3-touching set of 1-intersecting strings requiring 7 colors (b) A  $k$ -touching set of 1-intersecting strings requiring  $k + 2$  colors (here  $k = 4$ ).

Note that the constant  $c$  in Conjecture 3.3 is at least 4. Figure 5(a) depicts a 3-touching set of seven 1-intersecting strings, in which any two strings intersect. Hence, this set requires seven colors. However this construction does not extend to  $k$ -touching sets with  $k \geq 4$ , it might be that the constant is smaller for higher  $k$ . In Figure 5(b), the  $k$ -touching set  $\mathcal{S}_k$  contains  $k + 2$  non-crossing strings, and is such that any two strings intersect. Hence,  $k + 2$  colors are required in any proper coloring.

#### 4. 1-INTERSECTING CONTACT SYSTEM OF STRINGS

In this section, all the sets of strings we consider are 1-intersecting (any two strings intersect in at most one point). An interesting example of 1-intersecting set of strings is the family of non-crossing (straight-line) segments in the plane. This family is also known as *contact system of segments*,

and has been studied in [9], where the authors proved that any bipartite planar graph has a contact representation with horizontal and vertical segments.

More generally, a *contact system of strings* is a family of strings such that the interiors of the strings are pairwise non-intersecting. In other words, if  $c$  is a contact point in the interior of a string  $s$ , all the strings containing  $c$  distinct from  $s$  end at  $c$ . A contact point  $p$  is a *peak* if every string containing  $p$  has an end at  $p$ . Otherwise, that is if  $p$  is an interior point of a string  $s$  and an end for all the other strings containing  $p$ ,  $p$  is *flat*. A flat contact point  $p$  is *one-sided* if all the strings ending at  $p$  are on the same side of the unique string whose interior contains  $p$ . A contact system of strings in which every flat contact point is one-sided is also said to be one-sided.

It was proven by Ossona de Mendez [19] that the intersection graph of any one-sided 2- or 3-touching set of segments is planar. Note that as a 2-touching set of segments is always one-sided, it is also 4-colorable. Moreover, the author proved that it is NP-complete to determine whether a 2-touching set of segments is 3-colorable.

In [13], Hliněný studied the clique and chromatic numbers of one-sided  $k$ -touching contact systems of strings. He proved that the maximal clique in this class is  $K_{k+1}$  and that the graphs in this class are  $2k$ -colorable. He also asked the following: is there a constant  $c$  such that if a contact system of strings is  $k$ -touching, 1-intersecting, and one-sided, then it is  $(k + c)$ -colorable? Note that Conjecture 3.3 would imply a positive answer to this question.

In the first part of this section, we prove that 1-intersecting and  $k$ -touching contact systems of strings are  $(\lceil \frac{4}{3}k \rceil + 6)$ -colorable. In the second part, we show that any  $k$ -touching contact system of segments is  $(k + 5)$ -colorable. Note that we do not assume the contact systems to be one-sided (but we also show that adding this assumption slightly improves the additive constants in our results).

**Theorem 4.1.** *For any  $k \geq 3$ , any 1-intersecting  $k$ -touching contact system of strings can be colored with  $\lceil \frac{4}{3}k \rceil + 6$  colors.*

As in Theorem 2.2, the result is a consequence of a bound on the number of edges of these graphs:

**Lemma 4.2.** *For any  $k \geq 3$ , if  $\mathcal{S}$  is a 1-intersecting  $k$ -touching contact system of  $n$  strings, then  $G(\mathcal{S})$  has less than  $n(\frac{2}{3}k + 3)$  edges.*

*Proof.* Assume that there is a counterexample, i.e. a 1-intersecting  $k$ -touching contact system  $\mathcal{S}$  of  $n$  strings such that  $G(\mathcal{S})$  has  $m \geq n(\frac{2}{3}k + 3)$  edges. We take  $n$  minimal, and with respect to this,  $m$  maximal. Observe that  $G(\mathcal{S})$  is connected, since otherwise by minimality of  $n$ , each connected component would have average degree less than  $\frac{4}{3}k + 6$  and then we would have  $m < n(\frac{2}{3}k + 3)$ , a contradiction.

We now prove that every string of  $\mathcal{S}$  has at least two contact points. Assume that some string  $s$  has at most one contact point, then by the previous observation, it has a unique contact point  $p$ , which coincides with one of its ends. If every element of  $\mathcal{S}$  contains  $p$ , then  $G(\mathcal{S})$  is a clique on at most  $n \leq k$  vertices and has  $\binom{n}{2} < n \left(\frac{2}{3}k + 3\right)$  edges, a contradiction. Since  $G(\mathcal{S})$  is connected and is not a clique, it means that there is a string  $s_2$  at distance 2 from  $s$  in  $G(\mathcal{S})$ , i.e.,  $s_2$  does not contain  $p$  and touches some string  $s_1$  containing  $p$ . Take such a string  $s_2$  in such a way that the portion of  $s_1$  between  $p$  and  $s_1 \cap s_2$  does not contain any contact point. Then it is easy to redraw  $s$  as a string with ends  $p$  and some interior point of  $s_2$  close from  $s_1 \cap s_2$  (such that  $s$  contains no other intersection points than its two ends). This contradicts the maximality of  $m$ . This shows that every string of  $\mathcal{S}$  has at least two contact points. This also implies that the two ends of each string of  $\mathcal{S}$  are contact points (if not, delete the portion of a string between a free end and its first contact point).

Let  $H(\mathcal{S})$  be the plane graph whose vertices are the contact points of  $\mathcal{S}$ , whose edges link two contact points if and only if they are consecutive on a string of  $\mathcal{S}$ , and whose faces are the connected regions of  $\mathcal{P} \setminus \mathcal{S}$ .

Let  $p_i$  and  $f_i$  be the number of contact points of exactly  $i$  strings of  $\mathcal{S}$  that are respectively peaks and flat. Let us denote by  $c$  the total number of contact points, and note that  $c = \sum_{i=2}^k (p_i + f_i)$ . By counting the number of ends of a string of  $\mathcal{S}$  in two different ways, we obtain that:

$$(1) \quad 2n = \sum_{i=2}^k i p_i + \sum_{i=2}^k (i-1) f_i$$

Consider a one-sided flat contact point  $p$  and let  $s$  be the unique string such that  $p$  is an interior point of  $s$ . If we draw a small open disk  $D$  containing  $p$ , a unique face  $f$  of  $H(\mathcal{S})$  has the property that  $f \cap D$  is incident to  $s$ , and to no other string containing  $p$ . We denote this face  $f$  of  $H(\mathcal{S})$  by  $F(p)$ . Remark that since  $\mathcal{S}$  is 1-intersecting, any face  $f$  of  $H(\mathcal{S})$  contains at least  $|F^{-1}(f)| + 3$  vertices, thus at least  $|F^{-1}(f)|$  edges can be added to  $H(\mathcal{S})$  (inside  $f$ ) with the property that  $H(\mathcal{S})$  remains planar. Hence in total, one can add as many edges to  $H(\mathcal{S})$  as the number of one-sided contact points, while keeping  $H(\mathcal{S})$  planar. Since every flat 2-contact point is one-sided, and every planar graph on  $c$  vertices has at most  $3c - 6$  edges, it follows that  $H(\mathcal{S})$  has at most  $3c - 6 - f_2$  edges. As a consequence, we have:

$$\sum_{i=2}^k i p_i + \sum_{i=2}^k (i+1) f_i \leq 2 \cdot (3c - 6 - f_2).$$

This is equivalent to:

$$(2) \quad \sum_{i=2}^k (i-6)p_i + \sum_{i=2}^k (i-5)f_i \leq -12 - 2f_2$$

Since any pair of strings in  $\mathcal{S}$  intersects at most once, the number of edges in  $G(\mathcal{S})$  verifies the following equation.

$$(3) \quad m = \sum_{i=2}^k \binom{i}{2} (p_i + f_i)$$

Let us consider the linear program (LP1) defined on variables  $p_i$  and  $f_i$  with values in  $\mathbb{R}^+$  such that the equation (1) and the inequality (2) are verified, and such that the value  $m$  defined by (3) is maximized. Here  $n$  is considered to be a constant (it is not a variable of the linear program). Note that the solution  $m^*$  of this problem is clearly an upper bound of the number of edges of  $G(\mathcal{S})$ .

**Claim 4.3.** *The optimal solutions of (LP1) are such that  $f_2 = 0$ .*

If  $f_2 \neq 0$ , take a small  $\epsilon > 0$  and replace  $f_2$  by  $f_2 - \epsilon$  and  $f_3$  by  $f_3 + \epsilon/2$ . Then (1) remains valid, inequality (2) still holds (both sides are increased by  $2\epsilon$ ), while (3) is increased by  $\epsilon/2$ .

**Claim 4.4.** *The optimal solutions of (LP1) are such that  $f_i = 0$ , for every  $4 \leq i \leq k-1$ .*

If for some  $4 \leq i \leq k-1$ ,  $f_i \neq 0$ , choose a small  $\epsilon > 0$  and replace (i)  $f_3$  by  $f_3 + \epsilon \frac{(k-i)}{k-3}$ ; (ii)  $f_i$  by  $f_i - \epsilon$ ; and (iii)  $f_k$  by  $f_k + \epsilon \frac{i-3}{k-3}$ . Then (1) remains valid, inequality (2) still holds (the left-hand side and the right-hand side remain unchanged), while the value of  $m$  in (3) is increased by  $\frac{\epsilon}{k-3} (3(k-i) - \binom{i}{2}(k-3) + \binom{k}{2}(i-3))$ . The function  $g : i \mapsto 3(k-i) - \binom{i}{2}(k-3) + \binom{k}{2}(i-3)$  is a (concave) parabola with  $g(4) = \binom{k}{2} - 3k + 6 > 0$  (recall that  $4 \leq i \leq k-1$ , so  $k \geq 5$ ) and  $g(k) = 0$ , so it is positive in the interval  $[4, k-1]$ .

**Claim 4.5.** *The optimal solutions of (LP1) are such that  $p_i = 0$ , for every  $3 \leq i \leq k-1$ .*

If for some  $3 \leq i \leq k-1$ ,  $p_i \neq 0$ , choose a small  $\epsilon > 0$  and replace (i)  $p_2$  by  $p_2 + \epsilon \frac{(k-i)}{k-2}$ ; (ii)  $p_i$  by  $p_i - \epsilon$ ; and (iii)  $p_k$  by  $p_k + \epsilon \frac{i-2}{k-2}$ . Then (1) remains valid, inequality (2) still holds (the left-hand side and the right-hand side remain unchanged), while the value of  $m$  in (3) is increased by  $\frac{\epsilon}{k-2} ((k-i) - \binom{i}{2}(k-2) + \binom{k}{2}(i-2))$ . The function  $g : i \mapsto (k-i) - \binom{i}{2}(k-2) + \binom{k}{2}(i-2)$  is a (concave) parabola with  $g(3) = \binom{k}{2} - 2k + 3 > 0$  (recall that  $3 \leq i \leq k-1$ , so  $k \geq 4$ ) and  $g(k) = 0$ , so it is positive in the interval  $[3, k-1]$ .

It follows from the previous claims that  $c = p_2 + f_3 + p_k + f_k$ , so (2) gives  $-4p_2 + (k-6)p_k - 2f_3 + (k-5)f_k < 0$ . Therefore,

$$(k-6)(p_k + f_k) < 4p_2 + 2f_3.$$

By equation (1),

$$2p_2 + f_3 \leq 2p_2 + 2f_3 = 2n - kp_k - (k-1)f_k \leq 2n - (k-1)(p_k + f_k),$$

which implies  $(k-6)(p_k + f_k) < 4n - 2(k-1)(p_k + f_k)$ . This can be rewritten as

$$(3k-8)(p_k + f_k) < 4n.$$

Now, by equation (3),

$$\begin{aligned} m = p_2 + 3f_3 + \binom{k}{2}(p_k + f_k) &\leq \frac{3}{2}(2n - (k-1)(p_k + f_k)) + \binom{k}{2}(p_k + f_k) \\ &\leq 3n + (p_k + f_k)\left(-\frac{3}{2}(k-1) + \binom{k}{2}\right) \\ &< 3n + \frac{4n}{3k-8}(3k-8)\frac{k}{6} \quad (\text{since } k \geq 2) \\ &< n\left(\frac{2}{3}k + 3\right). \end{aligned}$$

The graph  $G(\mathcal{S})$  has less than  $n(\frac{2}{3}k + 3)$  edges, as desired.  $\square$

If the contact system we consider is one-sided, the argument we used for flat 2-intersection points while establishing (2) in the previous proof works for all flat points, and inequality (2) can be replaced by the stronger

$$(4) \quad \sum_{i=2}^k (i-6)p_i + \sum_{i=2}^k (i-3)f_i \leq -12$$

Let (LP2) be the new linear program. We can repeat the same argument as above and prove that in some optimal solution of (LP2), we also have  $f_3 = 0$ . Similar computations give that in this case the graph  $G(\mathcal{S})$  has less than  $n(\frac{2}{3}k + 1)$  edges. As a consequence:

**Theorem 4.6.** *For any  $k \geq 3$ , any 1-intersecting one-sided  $k$ -touching contact system of strings can be colored with  $\lceil \frac{4}{3}k \rceil + 2$  colors.*

We now show how to modify the proof of Theorem 4.1 to prove that  $k$ -touching contact systems of segments are  $(k+5)$ -colorable. For this we will need an additional idea, based on the notion of *stretchability*.

A contact system of strings is *stretchable* if there is a homeomorphism that transforms it into a contact system of segments. A *pseudo-line* is the homeomorphic image of a straight line. An *arrangement of pseudo-lines* is a set of pseudo-lines such that any two of them intersect at most once, and when they do they cross each other. We say that a contact system of strings is *extendible* if there is an arrangement of pseudo-lines, such that each string of the contact system is contained in a distinct pseudo-line of the arrangement. It was proved by de Fraysseix and P. Ossona de Mendez [10] that a contact system of strings is stretchable *if and only if* it is extendible.

We will use this equivalence to show the main result of this section.

**Theorem 4.7.** *For any  $k \geq 3$ , any  $k$ -touching contact system of segments can be properly colored with  $k + 5$  colors.*

As in Theorem 2.2, the result is a consequence of a bound on the number of edges in the corresponding intersection graphs:

**Lemma 4.8.** *For any  $k \geq 3$ , if  $\mathcal{S}$  is a  $k$ -touching contact system of  $n$  segments, then  $G(\mathcal{S})$  has less than  $\frac{1}{2}(k+5)n$  edges.*

*Proof.* The proof is similar to that of Theorem 4.1. We consider a counterexample  $\mathcal{S}$  consisting of  $n$   $k$ -touching segments such that  $G(\mathcal{S})$  has  $m \geq \frac{1}{2}(k+5)n$  edges. Again, we take  $n$  minimal, and with respect to this,  $m$  maximal. As before, we can assume that  $G(\mathcal{S})$  is connected and that the two ends of each segment are contact points.

We again consider the plane graph  $H(\mathcal{S})$  whose vertices are the contact points of  $\mathcal{S}$ , whose edges link two contact points if and only if they are consecutive on a segment of  $\mathcal{S}$ , and whose faces are the connected regions of  $\mathcal{P} \setminus \mathcal{S}$ . Let  $p_i$  and  $f_i$  be the number of contact points of exactly  $i$  segments of  $\mathcal{S}$  that are respectively peaks and flat. Let  $p = \sum_{i=2}^k p_i$  and  $c = \sum_{i=2}^k (p_i + f_i)$ .

Recall that any face  $f$  of  $H(\mathcal{S})$  contains at least  $|F^{-1}(f)| + 3$  vertices (where  $F^{-1}(f)$ , defined in the proof of Lemma 4.2, is the number of flat contact points “incident” to  $f$ ), thus at least  $|F^{-1}(f)|$  edges can be added to  $H(\mathcal{S})$  (inside  $f$ ) with the property that  $H(\mathcal{S})$  remains planar. We now show that, moreover, the vertices corresponding to the peaks of  $\mathcal{S}$  all lie on the outerface of  $H(\mathcal{S})$ . This directly implies that we can add  $f_2 + p - 3$  edges to  $H(\mathcal{S})$ , while keeping  $H(\mathcal{S})$  planar (as a consequence,  $H(\mathcal{S})$  has at most  $3c - 6 - (f_2 + p - 3) = 3c - 3 - f_2 - p$  edges).

Indeed, if some peak  $x$  of  $\mathcal{S}$  is not incident to the outerface, we choose a segment  $s$  containing  $x$  and prolong  $s$  after  $x$  until it hits some other segment  $s'$  (note that since  $s$  and  $s'$  are segments, they did not intersect previously). Let  $\mathcal{S}'$  be the new contact system of segments. By the maximality of  $m$ , the point  $x' = s \cap s'$  is a contact point containing  $k+1$  segments (including  $s$  and  $s'$ ) in  $\mathcal{S}'$ . To correct this, we do the following. By the result of de Fraysseix and P. Ossona de Mendez [10],  $\mathcal{S}'$  is extendible, so let  $\mathcal{L}$  be an arrangement of pseudo-lines extending  $\mathcal{S}'$  (i.e. such that every segment of  $\mathcal{S}'$  is contained in a different pseudo-line of  $\mathcal{L}$ ). Slightly perturb the pseudo-line  $\ell$  containing  $s$  around  $x'$ , so that it now intersects  $s'$  in some interior point of  $s'$ , very close from  $x'$ . The new arrangement  $\mathcal{L}'$  is still an arrangement of pseudo-lines, and it extends a contact system of segments  $\mathcal{S}''$  obtained from  $\mathcal{S}'$  by slightly moving the end of  $s$ , starting from  $x'$ , along  $s'$ . This new contact system  $\mathcal{S}''$  contradicts the maximality of  $m$ .

It follows that inequality (2) in the proof of Theorem 4.1 can be replaced by:

$$(5) \quad \sum_{i=2}^k (i-4)p_i + \sum_{i=2}^k (i-5)f_i \leq -6 - 2f_2$$

We consider the linear program (LP3) defined on variables  $p_i$  and  $f_i$  with values in  $\mathbb{R}^+$  such that the equation (1) and the inequality (5) are verified, and such that the value  $m$  defined by (3) is maximized.

Claim 4.3 (the optimal solutions of (LP3) are such that  $f_2 = 0$ ) and Claim 4.4 (the optimal solutions of (LP3) are such that  $f_i = 0$ , for every  $4 \leq i \leq k-1$ ) are still verified. Claim 4.5 is now replaced by the following stronger claim:

**Claim 4.9.** *The optimal solutions of (LP3) are such that  $p_i = 0$ , for every  $2 \leq i \leq k$ .*

If for some  $2 \leq i \leq k$ ,  $p_i \neq 0$ , choose a small  $\epsilon > 0$  and replace  $p_i$  by  $p_i - \epsilon$ , and  $f_i$  by  $f_i + \epsilon \frac{i}{i-1}$ . Then (1) remains valid, inequality (5) still holds (the left-hand side is decreased by  $\frac{4\epsilon}{i-1}$  and the right-hand side remains unchanged), while the value of  $m$  in (3) is increased by  $\frac{\epsilon i}{2}$ .

It follows from Claims 4.3, 4.4, and 4.9 that  $c = f_3 + f_k$ , so (5) gives  $(k-5)f_k < 2f_3$ . By equation (1),  $2f_3 = 2n - (k-1)f_k$ , which implies  $(k-5)f_k < 2n - (k-1)f_k$ . This can be rewritten as

$$(k-3)f_k < n.$$

Now, by equation (3),

$$\begin{aligned} m = 3f_3 + \binom{k}{2}f_k &\leq \frac{3}{2}(2n - (k-1)f_k) + \binom{k}{2}f_k \\ &\leq 3n + f_k \frac{(k-1)(k-3)}{2} \\ &< n\left(\frac{k-1}{2} + 3\right). \end{aligned}$$

The graph  $G(\mathcal{S})$  has less than  $\frac{n}{2}(k+5)$  edges, as desired.  $\square$

It follows that intersection graphs of  $k$ -touching segments are  $(k+4)$ -degenerate and then  $(k+5)$ -colorable. Note that the  $(k+4)$ -degeneracy may not be tight for every  $k$ . Indeed, we only know graphs that are not  $(k+3)$ -degenerate for  $k \leq 6$ . Those graphs are obtained in the following way for  $k = 6$ : First consider the segments in a straight-line drawing of a planar triangulation with minimum degree 5 and maximum degree 6, such that any degree five vertices is at distance at least two from the outerface, and such that any pair of degree five vertices are at distance at least three appart. Now let  $s_1, \dots, s_{30}$  be the segments whose ends respectively touch a 5- and a 6-contact points. Those are the only segments that touch less than 10 other segments (they only touch 9 of them). To make each of them touch one more segment, prolong successively the segments  $s_1, s_2$  and so on, by their end that is at the 6-contact point, until reaching another segment.

Figure 6 depicts  $k$ -touching sets of segments requiring  $k+2$  colors, for  $k = 2, 3$ . However it does not appear to be trivial to extend this construction for any  $k \geq 4$ . Note that Figure 6(b) also shows that there are intersection graphs of 3-contact representations of segments (with two-sided contact points) that are not planar.

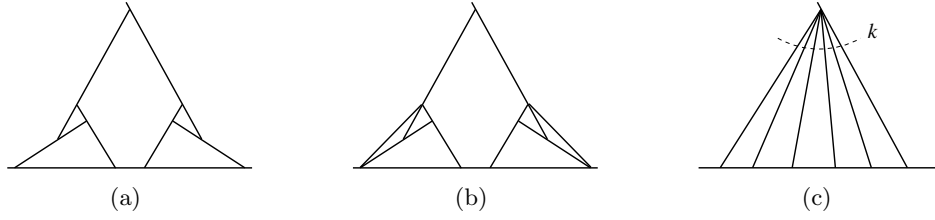


FIGURE 6. (a) A 2-touching set of segments requiring 4 colors (b) A 3-touching set of segments requiring 5 colors (c) A  $k$ -touching set of segments requiring  $k + 1$  colors.

If the contact system of segments we consider is one-sided, we can prove that in a counterexample maximizing  $m$ , all peaks are on the outface. Therefore, in this case inequality (4) in the proof of Theorem 4.6 can be replaced by the stronger

$$(6) \quad \sum_{i=2}^k (i-4)p_i + \sum_{i=2}^k (i-3)f_i \leq -6$$

From this, it is not difficult to derive the following small improvement over the previous result (we omit the details, since the proof is nearly the same).

**Theorem 4.10.** *For any one-sided  $k$ -touching contact system of  $n$  segments  $\mathcal{S}$ , the graph  $G(\mathcal{S})$  has less than  $\frac{1}{2}(k+4)n$  edges and is thus  $(k+4)$ -colorable.*

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